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Solving some types of ordinary differential equations by using Chebyshev derivatives direct residual spectral method

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ABSTRACT

Herein, novel basis orthogonal polynomials have developed. These developed polynomialshave been used to find the approximation solutions for some types of linear and non-linear ordinary differential equations by direct numerical method. This numerical method depends on the Chebyshev polynomials' derivatives. We shall present these solutions in the form of a finite sum of the Chebyshev polynomials' derivatives and unknown coefficients involving these polynomials. By substituting into the differential equation, the given differential equation will be converted into a system of algebraic equations. The obtained algebraic system can be solved easily to get the values of the spectral expansion constant. In addition, an algorithm for the approximated process has been designed to be easily used in the coding process. Consequently, some ordinary differential equations have been solved via the introduced Chebyshev polynomials' derivatives. Finally, the approximated solutions have been compared with exact and other methods solutions to illustrate the efficiency and accuracyof the used method.

1. Introduction

Ordinary differential equations (ODEs) underpin many applications in fields such as engineering, biology, and fluid dynamics [1–4], such that some of the problems in these fields and others can berepresented as ODEs. Many researchers use numerical methods for solving these equations such that analytical techniques cannot treat some problems. Numerical methods like spectral, finite element, and finite difference methods can give approximation solutions for many types of differentials and integrodifferential equations [5–8]. The approximation solution is semi-analytical using spectral methods, unlike finite difference and finite element methods. Spectral

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methods give accurate solutions to many types of differential and integral equations. The fundamental idea behind these methods is choosing suitable linear combinations of different special functions, often orthogonal polynomials.

The spectral method uses different types of orthogonal polynomials, which are called basis functions, such as Chebyshev polynomials [9, 10] or their derivatives [11], Legendre polynomials [12] or their derivatives [13–15], and Ultraspherical polynomials [16]. Spectral methods can solve ordinary differential equations by representing the unknown function in these equations as a finite series of well-known polynomials. This representation leads to an approximate solution. We can represent the solution as follows:

$$u(t) \approx u_n(t) = \sum_{k=0}^{n} a_k \emptyset_k(t),$$

where $\emptyset_k(t)$ represents the choice basis functions and a_k is a set of constants. After applying spectral methods, the differential equation will be converted to a system of algebraic equations with unknown constants. This system can be solved by any numerical techniques, such as the Gauss elimination method in linear systems and Newton Raphson's approximation for non-linear systems, to get the values of a_k that we can use. As a result, this set of constants is employed to get the approximate solution. Spectral methods categorically fall into three primary classes, namely, Galerkin, tau, and pseudo-spectral methods. The authors in [11–12,17–19] used tau and pseudo-Galerkin methods to solve higher-order ODE, while the authors in [20, 21] used the pseudo-spectral method. In this study, we will extend this approach by Chebyshev polynomials' derivatives as basis functions to improve the results.

This paper has been organized as follows: In the second section, the essential relations of Chebyshev polynomials will be presented. In the third section, the method used for finding the approximation solution will be discussed. Then, the linear and non-linear differential equations will be solved to show the proposed method's efficiency in the fourth section. Finally, the paper's concluding remarks were included in the fifth section.

2. Preliminaries

The essential concepts and relations for Chebyshev polynomials (CHPs) will be introduced. Consider that the Chebyshev polynomial is denoted by $T_j(t)$ which has degree j and $t\epsilon[-1,1]$.

CHPs are eigenfunctions for the Sturm-Liouville problem [22]:

$$(1-t^2)T_j''(t) - t\,T_j'(t) + j^2\,T_j(t) = 0, \quad t\epsilon[-1,1]. \quad (1)$$

The recurrence relations for CHPs are:

$$T_{j+1}(t) = 2t\,T_j(t) - T_{j-1}(t), \quad (2)$$

$$2T_j(t) = \frac{1}{j+1}T'_{j+1}(t) - \frac{1}{j-1}T'_{j-1}(t), \quad (3)$$

where $T_0(t) = 1, T_1(t) = t,$ and $j = 1,2,3,\ldots.$

CHPs and their derivatives satisfied the following relations:

$$|T_j(t)| \le 1, \quad (4)$$

$$|T'_j(t)| \le j^2, \quad (5)$$

$$T'_j(t) = \sum_{i=0}^{j-1} \frac{1}{c_i}\,2\,j\,T_i(t), \quad (i+j)odd, \quad (6)$$

$$T''_j(t) = \sum_{i=0}^{j-1} \frac{1}{c_i}(j+1)[(j+1)^2 - i^2]T_i(t), \quad (7)$$
$$(i+j)odd,$$

and $c_i = \begin{cases} 2, & i = 0, \\ 1, & elsewhere. \end{cases}$

CHPs and their derivatives have boundary values:

$$T_j(\pm 1) = (\pm 1)^j, \quad (8)$$

$$T'_j(\pm 1) = (\pm 1)^{j-1}\,j^2, \quad (9)$$

$$T''_j(\pm 1) = \frac{1}{3}(\pm 1)^j\,j^2(j^2-1). \quad (10)$$

CHPs can be presented in power series form as:

$$T_j(t) = \sum_{i=0}^{\lfloor j/2 \rfloor} (-1)^i\,2^{j-2i-1}\,\frac{j}{j-i}\binom{j-i}{i}\,t^{j-2i}, \quad (11)$$

such that, $\lfloor j/2 \rfloor$ is the integer part of j.

In the following section, we will present an explanation of the method used for finding approximation solutions to various types of ordinary differential equations.

3. The suggested method and problem formulation

Let the following is the form ODE:

$$f\left(a_r(t)y^{(r)}(t), a_{r-1}(t)y^{(r-1)}(t), a_{r-2}(t)y^{(r-2)}(t), \ldots, a_0(t)y(t)\right) = 0, \quad t\epsilon[-1,1], \quad (12)$$

with the initial and the boundary conditions:

$$\begin{cases} y(-1)=\beta_0, & y(1)=\gamma_0, \\ y'(-1)=\beta_1, & y'(1)=\gamma_1, \\ \vdots & \\ y^{(m)}(-1)=\beta_m, & y^{(m)}(1)=\gamma_m. \end{cases} \quad (13)$$

The set $\{a_i(t)\}_{i=0}^{r}$ is the real-valued function, $\{\beta_i\}_{i=0}^{m}$ and $\{\gamma_i\}_{i=0}^{m}$ are constants whose number is equal to the order of the ODE.

The following is the approximation solution to the given ODE:

$$\begin{aligned} y(t) &\approx y_n(t)=\sum_{k=0}^{n} a_k T''_{k+2}(t), \\ y'(t) &\approx y'_n(t)=\sum_{k=0}^{n} a_k T'''_{k+2}(t), \\ &\vdots \\ y^{(r)}(t) &\approx y_n^{(r)}(t)=\sum_{k=0}^{n} a_k T_{k+2}^{(r+2)}(t), \end{aligned} \quad (14)$$

since a_k are constants.

Eqs. (14) will be applied to Eq. (12) and conditions in Eq. (13), to get the results:

$$f\left(\begin{array}{c} a_r(t)\sum_{k=0}^{n} a_k T_{k+2}^{(r+2)}(t), a_{r-1}(t)\sum_{k=0}^{n} a_k T_{k+2}^{(r+1)}(t), \\ ,\dots, a_0(t)\sum_{k=0}^{n} a_k T''_{k+2}(t) \end{array} \right)=0, \quad (15)$$

where $-1 \le t \le 1$ and the conditions:

$$\begin{cases} \sum_{k=0}^{n} a_k T''_{k+2}(-1)=\beta_0, & \sum_{k=0}^{n} a_k T''_{k+2}(1)=\gamma_0, \\ \sum_{k=0}^{n} a_k T'''_{k+2}(-1)=\beta_1, & \sum_{k=0}^{n} a_k T'''_{k+2}(1)=\gamma_1, \\ \vdots & \\ \sum_{k=0}^{n} a_k T_{k+2}^{(m+2)}(-1)=\beta_m, & \sum_{k=0}^{n} a_k T_{k+2}^{(m+2)}(1)=\gamma_m, \end{cases} \quad (16)$$

Eqs. (15) and (16) yield a system of equations characterized by unknown coefficients. These coefficients will subsequently be determined through the application of a numerical method and using the Mathematica program. As a result, the approximate solution will rely on the derivatives of Chebyshev polynomials.

The following algorithm shows the steps of the solution:

Algorithm 1 Algorithm Steps for Approximating ODE by second derivative CHP

Step 1: Enter $n \in \mathbb{N}$,
Step 2: The independent variable will be shifted from a defined domain to $[-1, 1]$
Step 3: Choose the collocation points.
Step 4: Substitute into the ODE (15-16).
Step 5: Solve the system from step 4 to find a_k
Step 6: Use the finding constants from step 5 to obtain the approximation solution.

4. Numerical examples

In this section, we will work on solving four examples to showcase how well the proposed method works in terms of accuracy and efficiency. These examples involve the Lane-Emden equations, the fourth-order differential equation, and the Bratu equation.

Example 4.1. *Consider the Lane-Emden equation, which is non-homogeneous as follows [23]:*

$$y''(t)+\frac{8}{t}y'(t)+t\,y(t)=t^5-t^4+44t^2-30t \quad (17)$$

where $0<t<1$ *with conditions* $y(0)=0$, $y'(0)=0$, *and the exact solution is* $y(t)=t^4-t^3$. *After applying our method for solving example (4.1) and shifting the domain from* $(0,1)$ *to* $(-1,1)$, *from Eq. (14), we have:*

$$\begin{aligned} y(t) \approx y_4(t) &= \sum_{k=0}^{4} a_k T''_{k+2}(t) \\ &= a_0T''_2(t)+a_1T''_3(t)+a_2T''_4(t) \\ &\quad + a_3T''_5(t)+a_4T''_6(t), \end{aligned}$$

with algebraic system:

$$24a_1-192a_2+840a_3-2688a_4=0 \quad (18)$$

$$4a_0-24a_1+80a_2-200a_3+420a_4=0 \quad (19)$$

$$a_0+1533a_1-5374a_2+3845a_3+13044a_4=-\frac{4867}{1024} \quad (20)$$

$$a_0+384a_1+380a_2-1920a_3-2295a_4=-\frac{129}{64} \quad (21)$$

$$3a_0+521a_1+2822a_2+6385a_3+4828a_4=\frac{2223}{1024} \quad (22)$$

The solution of this system is: $a_0 = -\frac{15}{1024}$, $a_1 = -\frac{5}{1536}$, $a_2 = a_3 = \frac{1}{2560}$, $a_4 = \frac{1}{15360}$.

So,

$$y_4(t) = \frac{-15}{1024}(4) + \frac{-5}{1536}(24t) + \frac{1}{2560}(96t^2 - 16) + \frac{1}{2560}(320t^3 - 120t) + \frac{1}{15360}(960t^4 - 576t^2 + 36) = \left(\frac{1+t}{2}\right)^4 - \left(\frac{1+t}{2}\right)^3.$$

This is equivalent to the exact solution for $t \in (-1,1)$ *at* $n = 4$.

Compared with other methods, the maximum absolute error was e-07 at $n = 30$ *in [24] and e-11 at* $n = 8$ *in [25]. In contrast, the approximation solution by our method was equivalent to the exact solution at small n. This confirms that our method is more accurate and efficient.*

Table 1: Point-wise absolute error for example (4.2)

t	**Suggested method** $n = 14$	**[26]** $n = 14$	**[27]** $n = 14$	**[28]** $n = 14$
0.0	2.44e-13	5.79e-12	-	6.72e-08
0.1	1.57e-13	3.60e-12	3.14e-10	6.69e-08
0.2	1.18e-13	2.61e-12	3.07e-10	7.87e-09
0.3	9.46e-14	2.01e-12	2.99e-10	6.92e-09
0.4	7.66e-14	1.57e-12	2.88e-10	2.87e-08
0.5	6.17e-14	1.21e-12	2.82e-10	7.40e-10
0.6	4.92e-14	8.93e-13	2.14e-10	6.32e-08
0.7	3.77e-14	6.22e-13	1.51e-10	6.95e-08
0.8	2.75e-14	3.77e-13	9.45e-11	3.38e-09
0.9	1.82e-14	1.62e-13	7.35e-11	7.85e-08
1.0	4.08e-17	8.69e-17	-	6.63e-08

Example 4.2. *Consider the Lane-Emden equation, which is non-linear as follows [26–28]*

$$y''(t) + \frac{1}{t}y'(t) + e^{y(t)} = 0 \quad (23)$$

where $0 < t < 1$ *with conditions* $y'(0) = 0, y(1) = 0$, *and the exact solution is* $y(t) = 2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})\,t^2+1}\right)$.

Table(1) shows the point-wise absolute error at $n = 14$ *for the interval* $[0, 1]$. *Also, Figure (1) and Figure (2) show the point-wise absolute error at* $n = 14$ *and* $n = 16$, *while Figure (3) comparesthe approximation solution with the exact solution at* $n = 16$.

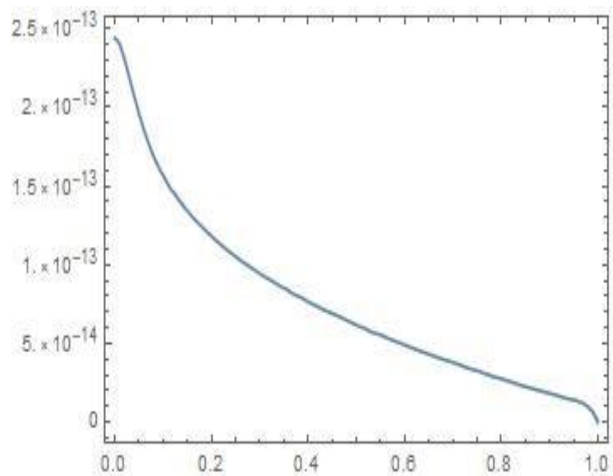

Figure 1: Point-wise absolute error for example (4.2) at $n = 14$.

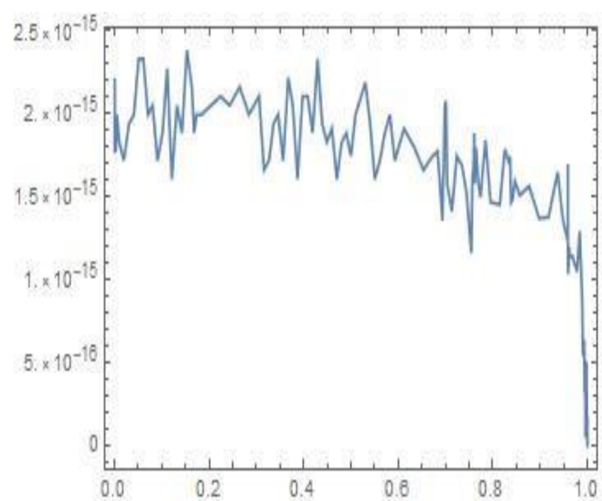

Figure 2: Point-wise absolute error for example (4.2) at n=16.

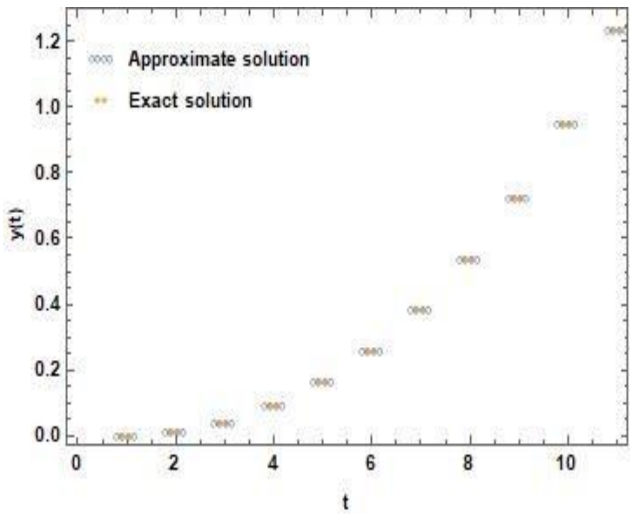

Figure 3: A comparison of the approximate and exact solutions for example (4.2) at $n = 16$.

Table 2: Point-wise absolute error for example (4.3).

T	Suggested method	[29]	[30]	[31]
0.0	3.63e-17	-	-	-
0.1	1.87e-08	4.20e-08	1.78e-07	2.99e-04
0.2	4.06e-08	1.72e-07	4.51e-07	0
0.3	6.33e-08	4.05e-07	7.19e-07	1.69e-04
0.4	8.74e-08	7.65e-07	1.01e-06	1.11e-04
0.5	1.14e-07	1.34e-07	1.32e-06	0
0.6	1.43e-07	2.07e-06	1.67e-06	0
0.7	1.76e-07	3.20e-06	2.06e-06	7.77e-05
0.8	2.16e-07	4.88e-06	2.06e-06	0
0.9	2.64e-07	7.36e-06	3.12e-06	3.47e-03
1.0	3.19e-07	-	-	-

Example 4.3. *Consider the Bratu equation as follows [29]:*

$$y''(t) - 2e^{y(t)} = 0, \qquad 0 \le t \le 1, \qquad (24)$$

with conditions $y(0) = 0,\ y'(0) = 0$ *and the exact solution is* $y(t) = 2\ln(\cos t)$.
Table (2) shows the point-wise absolute error for the interval $[0, 1]$. *Figure (4) compares the approximation solution with the exact solution at* $n = 16$.

Table 3: Point-wise absolute error for example (4.4).

t	Suggested method		[32]
	$n = 13$	$n = 16$	$n = 14$
-1	5.95e-14	2.22e-16	0
-0.6	6.74e-14	3.33e-16	2.17e-14
-0.2	6.67e-14	4.44e-16	3.46e-14
0.2	5.42e-14	3.89e-16	5.28e-14
0.6	2.72e-14	4.44e-16	1.23e-14
1.0	1.87e-15	4.62e-16	0

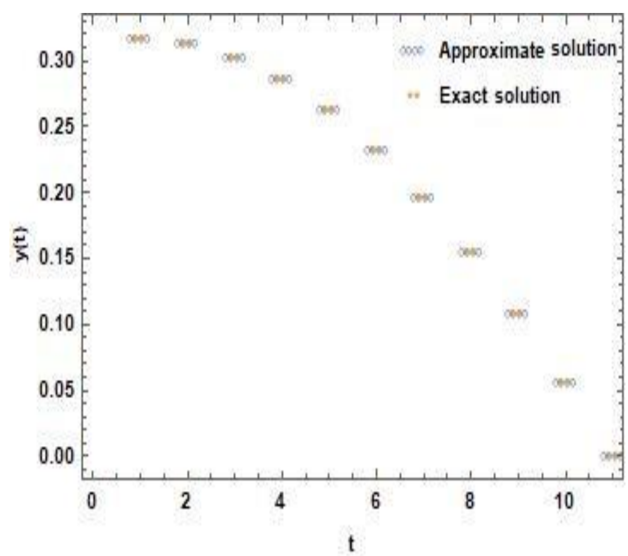

Figure 4: A comparison of the approximate and exact solutions for example (4.3) at $n = 16$.

Example 4.4. *Consider the fourth-order one-dimensional equation [32]:*

$$32\, y^{(4)}(t) - 8y^{(2)}(t) - 2y(t) = (1-t)e^{\frac{1+t}{2}}, \quad t \in (-1,1) \qquad (25)$$

with $y(-1) = 1,\ y'(-1) = 0,\ y(1) = 0,\ y'(1) - \frac{e}{2}$ *and*

the exact solution is $y(t) = \frac{(1-t)}{2} e^{\frac{1+t}{2}}$.
Table(3) shows the point-wise absolute error at $n = 14, n = 16$ *for the interval* $[-1, 1]$. *Also, Figure (5) shows the point-wise absolute error at n* = 16, *which proves the efficiency and accuracy of our method.*

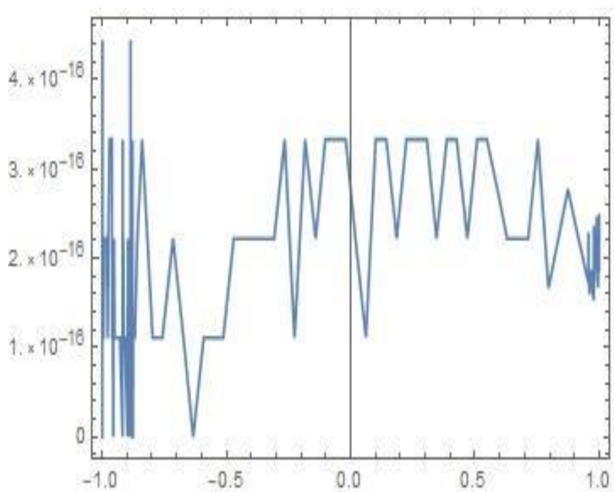

Figure 5: Point-wise absolute error for example (4.4) at *n* = 16.

Conclusions

This paper explores a novel trial function for solving both linear and non-linear ordinary differential equations using the spectral expansion method. Then, the suggested method presents solutions in the form of a finite sum of the Chebyshev polynomials' derivatives and unknown coefficients. Also, examples such as the Lane-Emden problem, the Bratu equation, and the fourth-order differential equation are solved to demonstrate the effectiveness of the suggested method.

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